Short Papers

On The Use of Davidenko's Method in Complex Root Search

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Abstract—Davidenko's method has proved to be a powerful technique for solving a system of n-coupled nonlinear algebraic equations. It employs a Newton's method reduction to produce n-coupled first-order differential equations in a dummy variable. The advantage it offers over Newton's method and other traditional methods such as Muller's method is that it relaxes the restrictions that the initial guess has to be very close to the solution. Two examples involving the search for complex roots are presented. Davidenko's method seems to converge to the roots for all the arbitrary initial guesses considered while Muller's method appears to fail for some cases. This suggests the use of Davidenko's method as an alternative to Muller's method when the later fails to converge or is slowly convergent.

I. Summary of Complex Root Search Methods

In many electromagnetics problems, such as microstrip antennas in multilayered dielectric structures, we often obtain complex transcendental expressions whose complex poles or singularities have to be found and extracted in order to accelerate the convergence of the numerical algorithm used. Rootfinding methods such as Newton–Raphson and Muller's methods have been the most commonly used by researchers. However, the success of these methods mainly depends on the proper choice of initial guesses. In this paper we will compare these methods to an alternative method, known as Davidenko's method, as applied to some typical electromagnetics applications.

The Newton–Raphson method [1] is based on the idea of approximating a complex function $f(z)$ locally by a straight line (tangent line). Let z_k be an approximation of root z_r. Then the equation of the tangent line to $f(z)$ at (z_k, f_k) is given by

$$f(z) - f_k = (z - z_k)f'_k \tag{1}$$

where f'_k is the derivative of $f(z)$ at $z = z_k$, and the z-intercept is $(z_{k+1}, 0)$ where

$$z_{k+1} = z_k - \frac{f_k}{f'_k}. \tag{2}$$

In regions where $f(z)$ has a small slope, Newton–Raphson method can be erratic. Direct convergence to a root z_r is guaranteed only if the initial guess z_o is sufficiently close to z_r.

Muller's method [2] is widely used by electromagnetics researchers in complex root search. It is better than Newton-Raphson method in the sense that it interpolates $f(z)$ in the vicinity of a root by a quadratic curve (parabola). Since it takes 3 points to uniquely define a parabola, we assume 3 approximate values for the root z_r, z_{k-2}, z_{k-1}, and z_k along with their respective function values f_{k-2}, f_{k-1}, and f_k. Therefore, given 3 initial guesses z_o, z_1, z_2, the method generates a sequence $z_o, z_1, z_2, z_3, \cdots$ of approximations. At each stage, the new approximation is derived from the 3 most recent ones by quadratic interpolation. Although the method is rather complicated, no evaluations of derivatives of $f(z)$ and only one evaluation of $f(z)$ is required per iteration. Convergence is essentially global for almost all practical problems with almost any set of initial guesses z_o, z_1, z_2.

An alternative method that is competitive to Muller's method is Davidenko's method. This method has been successfully applied in Chemical engineering problems [3]. It has also found use in Electromagnetics, where it was used in solving the dispersion relations of electromagnetic waves propagating in lossy waveguide structures [4]. Davidenko's method is a Newton-based method which is advantageous in the case of n-dimensional systems of nonlinear equations ($n \geq 2$), and hence may prove very useful when dealing with antenna problems in multilayered media. Consider the case of a complex transcendental equation of the form

$$F(z = x + jy) = 0 \tag{3}$$

where x and y represent the real and imaginary parts of the complex variable z. $F(z)$ is transformed, through the use of the Jacobian matrix, into a set of two nonlinear first-order ordinary differential equations in terms of the real and imaginary parts of the complex root, of the form [3]

$$\begin{aligned}\frac{dx}{dt} &= -\frac{1}{|F_z|^2}(\mathrm{Re}\,[F]\,\mathrm{Re}\,[F_z] + \mathrm{Im}\,[F]\,\mathrm{Im}\,[F_z])\\ \frac{dy}{dt} &= +\frac{1}{|F_z|^2}(\mathrm{Re}\,[F]\,\mathrm{Im}\,[F_z] - \mathrm{Im}\,[F]\,\mathrm{Re}\,[F_z])\end{aligned} \tag{4}$$

where $F_z = \partial F/\partial z$ is the partial derivative of $F(z)$. Since the solution to the ODE's in (4) is a decaying time exponential, the solution is reached when t is very large ($t \to \infty$). This method evaluates $F(z)$ and $F_z(z)$ in each iteration. Therefore, it requires that the complex function and its derivative be smooth in the interval of interest except at some finite number of points (in the Cauchy-sense). The advantage of Davidenko's method is that it relaxes the restriction on the choice of the initial guess for the complex root, and it can be used when other methods like Newton–Raphson and Muller fail to converge. It is important to remember that this method can not be used if an analytical expression for the complex function is not available.

II. Example of Finding TM and TE Surface-Wave Modes in Microstrip Antennas

In microstrip antennas, we often have to obtain the complex TE and TM surface-wave modes by finding the complex roots of the transcendental characteristic equations (D_{TE} and D_{TM}). The most commonly used methods are those of Newton–Raphson and Muller. Here, we will attempt to obtain the roots with Davidenko's method and compare our results with the other two methods.

The characteristic equations for the normalized TM and TE surface wave modes of the two-layer microstrip antenna shown in Fig. 1 are given by [6]

$$\begin{aligned}D_{\mathrm{TM}}^{(2)} &= \epsilon_{r2} u_1^2 \tanh(u_1 k_o h_1) + \epsilon_{r1} u_1 u_2\\ &\quad \cdot \tanh(u_2 k_o h_2) + \epsilon_{r1}\epsilon_{r2} u_o u_1 + \epsilon_{r1}^2 u_o u_2\\ &\quad \cdot \tanh(u_1 k_o h_1)\tanh(u_2 k_o h_2) = 0\end{aligned} \tag{5}$$

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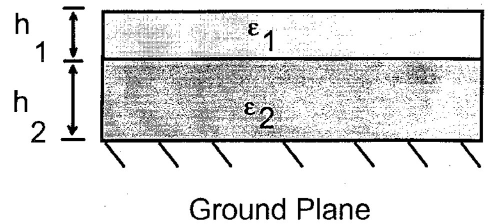

Fig. 1. A double-layer microstrip antenna.

and

$$D_{\rm TE}^{(2)} = u_1u_2 + u_ou_2 \tanh(u_1k_oh_1) + u_ou_1 \tanh(u_2k_oh_2) + u_1^2 \tanh(u_1k_oh_1) \cdot \tanh(u_2k_oh_2) = 0 \quad (6)$$

where

$$u_i = \sqrt{z^2 - \epsilon_{r_i}}, \quad z = x + jy = \frac{k_\rho}{k_o} \quad (7)$$

where $\epsilon_{r_i} = \epsilon'_{r_i} - j\epsilon_{r_i}$ is the complex dielectric constant of the ith-dielectric layer and h_i its thickness; k_o is the free-space wavenumber and k_ρ the spectral ρ-component of the propagation constant. Mosig [5] has shown that the roots of $D_{\rm TM}^{(2)}$ and $D_{\rm TE}^{(2)}$ exist only for $x \in [1, \max\sqrt{\epsilon_{r_i}}]$. The expressions in (5) and (6) reduce to those for an open microstrip antenna if layer 1 is assumed to be an air dielectric ($\epsilon_{r_1} = \epsilon_{r_o} = 1, h_1 \to \infty$). Hence, for an open microstrip the TM and TE characteristic equations become

$$D_{\rm TM}^{(1)} = \epsilon_{r_2}u_o + u_2 \tanh(u_2h_2) = 0 \quad (8)$$

and

$$D_{\rm TE}^{(1)} = u_o + u_2 \coth(u_2h_2) = 0. \quad (9)$$

For the open microstrip (single layer), Mosig [5] has shown that $D_{\rm TM}^{(1)}$ has only one complex zero in the range $1 < x < \sqrt{\epsilon_{r_2}}$ if $k_oh_2\sqrt{\epsilon_{r_2} - 1} < \pi/2$ while $D_{\rm TE}^{(1)}$ has no zeros. First derivatives of expressions (5), (6), (8) and (9) exist except at finite number of points (branch point at $k_\rho = k_o$ and infinite derivative at $k_\rho = k_o\sqrt{\epsilon_{r_i}}$) and hence the application of Davidenko's method will be possible.

For numerical computations, consider an open microstrip antenna of substrate thickness $k_oh_2 = 0.6384$ and relative complex dielectric permittivity $\epsilon_{r_2} = 2 - j\ 0.01$ (Teflon). Using a sketch of the magnitude of $D_{\rm TM}^{(1)}$ in the complex plane $z = x + jy\,(y < 0)$ it can be determined that the root lies in the region $\mathrm{Re}(z) \in [1, 1.1]$ for the given data. Table I shows the number of iterations required by Davidenko's method in comparison to the other methods. The iterative process in Davidenko's method is stopped when the absolute difference between two successive iterations $|z_{k+1} - z_k| \le \delta\,(\delta = 10^{-6})$. The root is found to be approximately at $z_p = (1.051756, -6.205234E-4)$. Note that all the methods considered converge to the exact root when the initial guess is nearly real. Newton–Raphson (A) fails if the initial guess is picked far away from the root location (in this case for Re $(z) > 4.0$). It is interesting to note that Davidenko's method (C) requires the same number of iterations ($N \simeq 15$) to converge to the root independently of the initial guess assumed. On the other hand, in Muller's method (B) the number of iterations varies at random, and it can fail for some initial guesses. Also, for some guesses, Davidenko's method appears to converge faster.

The roots of $D_{\rm TM}^{(2)}$ and $D_{\rm TE}^{(2)}$ were also computed using Muller's and Davidenko's methods. We have chosen GaAs ($\epsilon_{r_1} = 12.5$) with a thickness of $k_oh_1 = 0.1885$ and Teflon ($\epsilon_{r_2} = 2.1$) with a thickness of $k_oh_2 = 0.4398$ as the superstrate and substrate materials, respectively. Two surface-wave poles have been found for

TABLE I

NUMBER OF ITERATIONS NEEDED BY A) NEWTON –RAPHSON' S, B) MULLER'S AND C) DAVIDENKO' S METHODS TO FIND THE ROOTS OF $D_{\rm TM}^{(1)}$ IN (8)

Guess	A	B	C
(1.1,0)	3	7	11
(2,-2)	7	10	14
(4,-4)	Fails	Fails	14
(6,-9)	Fails	16	15
(5,-7)	Fails	25	15
(10,1)	Fails	23	15
(3,-1)	7	12	14
(10,-8)	Fails	16	15
(3,-5)	10	12	14
(3,-4)	9	Fails	14

this geometry, one for TM at $z_p^{\rm TM} = 1.175291$ and the other for TE at $z_p^{\rm TE} = 1.033629$. Both methods converged to the roots with the selected initial guesses and no divergence was noted. Davidenko's method again requires a consistent number of iterations to converge to the root. However, the number of iterations required by Muller's method varies greatly with the initial guess, and can be greater than that required by Davidenko's method.

III. EXAMPLE OF FINDING THE COMPLEX DIELECTRICCONSTANT FROM REFLECTION COEFFICIENT MEASUREMENTS

The calculation of the complex relative dielectric permittivity from reflection coefficient measurements of a dielectric material sample embedded in a rectangular waveguide is well known and the theory is fully established [7]. In this example, the sample is embedded at one end of the guide (filling part of the guide) and backed by a matched load termination. The other end of the guide is connected to an HP 8510B Network Analyzer as shown in Fig. 2. Assuming that only the $\rm TE_{10}$ mode propagates, we use the theory of wave propagation in multilayered media to determine the reflection coefficient at the input of the guide (S_{11}). Following [7] we obtain the reflection coefficient at the plane of the sample (plane T_S) as

$$\Gamma_S = \frac{-j(Z_o^2 - Z_1^2)\sin(\beta_1t_o)}{2Z_oZ_1\cos(\beta_1t_o) + j(Z_o^2 + Z_1^2)\sin(\beta_1t_o)} \quad (10)$$

where $Z_i = \omega\mu/\beta_i$ is the $\rm TE_{10}$ characteristic impedance of medium $i\,(i = 0$ in air, $i = 1$ in dielectric), $\beta_i = \sqrt{k_i^2 - k_c^2}$ its phase constant and $k_i = k_o\sqrt{\epsilon_{r_i}}$ its wavenumber. $k_c = \pi/a$ is the cutoff wavenumber of the guide with a being the guide widest dimension. The measured S_{11} parameter (at plane T in Fig. 2) is related to Γ_S by the phase shift relation

$$S_{11} = \Gamma_S e^{j2\beta_o(l_o - t_o)} \quad (11)$$

where l_o is the guide length ($l_o = 5$ inches for X-band waveguide). The only unknown in (10) is the complex relative dielectric permittivity which will be obtained by transforming (10) into a complex transcendental equation of the form

$$F(\epsilon_r) = 2\Gamma_S z\cos(\beta_1t_o) + j((1 + \Gamma_S)z^2 - (1 - \Gamma_S))\sin(\beta_1t_o) = 0 \quad (12)$$

where $z = \beta_1/\beta_o$ is a function of ϵ_r. The derivative of $F(\epsilon_r)$ exists and hence Davidenko's method is applicable.

Numerical computations of the complex roots of (12) were performed at an X-band frequency of 8.305 GHz for a 0.255- thick Teflon sample. The measured S_{11} parameter was $0.5693\angle 155.89°$. By plotting the magnitude of $F(\epsilon_r)$ versus the real and imaginary parts of ϵ_r it can be determined that the function has a root whose real part is in the proximity of 2. Table II shows the number of

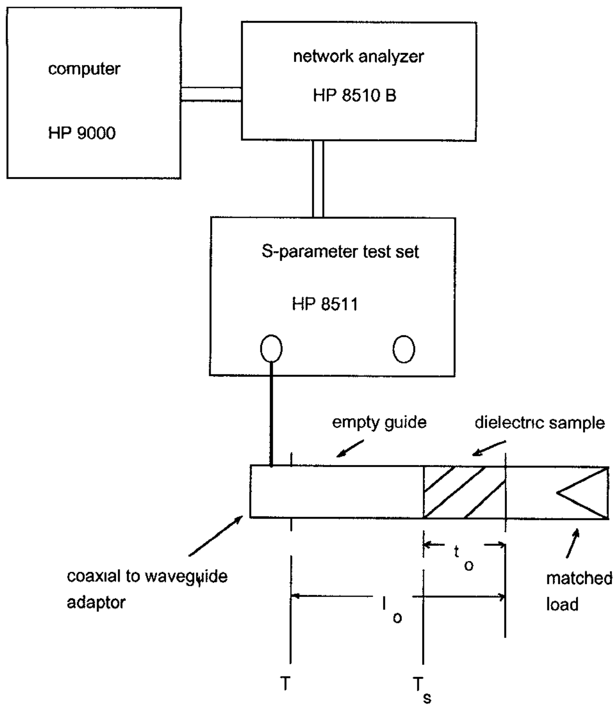

Fig. 2. HP 8510B automated network analyzer set-up for the measurement of the complex reflection coefficient of a dielectric sample embedded in a matched rectangular waveguide.

TABLE II
NUMBER OF ITERATIONS NEEDED BY A) MULLER' S AND B) DAVIDENKO' S METHODS TO FIND THE COMPLEX DIELECTRIC CONSTANT FROM REFLECTION COEFFICIENT MEASUREMENTS AND (12)

Guess	A	B
(3,-1)	7	15
(1,-3)	11	16
(4,-1)	7	15
(5,-4)	9	16
(2,-1)	7	15
(5,-1)	8	16
(1,3)	Fails	17
(5,4)	9	17
(1,6)	Fails	18
(2,10)	13	18

iterations required by each method to converge to the root within a specified tolerance (10^{-6}). Only Muller's and Davidenko's methods were compared. In this example, Muller's method appears to converge faster than Davidenko's for some initial guesses but diverges for other values. As in the previous example, the number of iterations required by Davidenko's method to converge is independent of the initial guess chosen ($N \simeq 17$). For the given measured data, both Muller's and Davidenko's methods yield a complex relative dielectric permittivity of $\epsilon_r \simeq (2.080465, -0.051842)$.

IV. CONCLUSIONS

In this paper we explored the capabilities of Davidenko's method as a complex root-search routine. It shows to be as promising as Muller's method and hence could be used as an alternative if Muller's method is slowly convergent or if it fails to converge to the root. The only apparent setback for Davidenko's method is that it requires the analytical expression of the first derivative (if it exists) of the complex function.

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Mutual Coupling Between Two Small Circular Apertures in a Conducting Screen

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***Abstract*—With the use of the reaction integral and two-dimensional Fourier transform, an analytical expression for mutual coupling between two small circular apertures in a conducting screen, excited by normally incident plane electromagnetic wave, is obtained. Numerical examples for two different polarizations of the plane wave are investigated. The expression for the mutual admittance gives a correct value of the self admittance of a small aperture when the distance between the holes is equal to zero.**

I. INTRODUCTION

The problem of computing the mutual coupling between two equal apertures is a classical one. For the case of two narrow parallel rectangular apertures excited by a plane electromagnetic wave, it is dual of the problem of computing mutual coupling between two electric dipoles, which was solved for the first time by Carter [1] and later more accurately by King [2]. The problem of computing analytically the mutual coupling between two open circular waveguides was solved by Bailey [3]. The problem of the computing analytically the mutual coupling between two circular apertures in a conducting screen, excited by a plane wave, is studied in this paper.

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